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THE EVALUATION OF QUANTUM INTEGRALS

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The application of the Wilson-Sommerfeld quantum conditions to a conditionally periodic system with orthogonal coördinates involves the evaluation of an integral of the type

$$J = \oint \sqrt{f(q)}\, dq. \tag{1}$$

The integral is to be extended over a complete cycle of values of q, which oscillates between two roots of $f(q)$. The sign of the radical is to be the same as that of dq, so that if a and b denote the roots of $f(q)$, the integral can be written

$$J = 2\int_a^b \sqrt{f(q)}\, dq \tag{2}$$

If $f(q)$ is a polynomial of the second degree in either q or $1/q$ the integral can be cleanly evaluated. Otherwise, approximations are generally necessary. If $f(q)$ can be expressed in the form

$$f(q) = \varphi(q) + \alpha\psi(q)$$

where $\varphi(q)$ is quadratic in q or $1/q$, α is constant, and $\alpha\psi(q)$ is small, a natural method of procedure is to try to develop J into a power series in α. Thus

$$J(\alpha) = J(0) + \alpha J'(0) + \frac{\alpha^2 J''(0)}{2!} + \frac{\alpha^3 J'''(0)}{3!} + \cdots \tag{3}$$

$J(0)$ and $J'(0)$ are easily evaluated, but unfortunately the higher derivatives of J with respect to α cannot be calculated by the usual methods because the higher derivatives of $\sqrt{(fq)}$ with respect to α become infinite at $q = a$ and $q = b$. Hence this method is useful only when the higher order terms are negligible.[1]

Another method of attack employed by F. Tank[2] and accepted as valid by others[3] turns out on close examination to be faulty. Tank de-

velops $f(q)$ into a power series about its maximum point, $q = d$. Let $\xi = q - d$, and let H denote the maximum value of $f(q)$. Then $f(q)$ can be thrown into the form

$$f(q) = H - \alpha\xi^2 - (\beta\xi^3 + \gamma\xi^4 + \delta\xi^5 + \ldots). \tag{4}$$

Introducing the symbols

$$V(\xi) = \alpha\xi^2,$$
$$\Delta V = \beta\xi^3 + \gamma\xi^4 + \delta\xi^5 + \ldots,$$

we obtain

$$\sqrt{f(q)} = \sqrt{H-V}\sqrt{1-\frac{\Delta V}{H-V}} = \sqrt{H-V} - \frac{1}{2}\frac{\Delta V}{\sqrt{H-V}} - \frac{1}{8}\frac{(\Delta V)^2}{\sqrt{(H-V)^3}} - \ldots$$

Tank in effect integrates this series term by term between the limits $-(H/\alpha)^{1/2}$ and $+(H/\alpha)^{1/2}$ and identifies twice the sum of the series so obtained with J. This procedure is wrong, since the correct limits of integration for J are $\xi = a - d$ and $\xi = b - d$. Moreover, the expansion is not usually convergent throughout the interval of integration, so that it is not possible to correct Tank's work by altering the limits.

Another scheme of series development may be suggested, which avoids the above difficulty. Let the quantity u be defined by the equation

$$H = \sqrt{H - f(q)} = \xi\sqrt{\alpha + \beta\xi + \gamma\xi^2 + \delta\xi^3 + \ldots}. \tag{5}$$

The sign of u is to be the same as that of ξ. The integral J can now be thrown into the form

$$J = 2\int_{u=-\sqrt{H}}^{u=+\sqrt{H}} \sqrt{H-u^2}\,\frac{d\xi}{du}\,du \tag{6}$$

Let us assume that $d\xi/du$ can be developed into a power series in u. Thus

$$\frac{d\xi}{du} = \sum_{r=0}^{\infty} a_r u^r \tag{7}$$

Let

$$K_r = 2\int_{-\sqrt{H}}^{+\sqrt{H}} u^r\sqrt{H-u^2}\,du. \tag{8}$$

To evaluate K_r we introduce the variable of integration θ defined by the relation $u = H^{1/2}\sin\theta$.

Then (8) becomes

$$K_r = H^{\frac{r+2}{2}}\int_0^{\pi} \sin^r\theta\cos^2\theta\,d\theta. \tag{9}$$

It is easy to show from (9) by the application of well-known formulae that K_τ vanishes for odd values of τ; that K_0 is πH; and that for even values of τ greater than zero K_τ is given by the equation

$$K_\tau = 2\pi H^{\frac{\tau+2}{2}} \varphi(\tau) \tag{10}$$

where

$$\varphi(\tau) = \frac{1.3.5.\ldots.(\tau-1)}{2.4.6.\ldots.(\tau+2)}. \tag{11}$$

Now (6) becomes

$$J = \pi H [a_0 + 2 \sum_{\tau=1}^{\infty} a_{2\tau} \varphi(2\tau) H^\tau]. \tag{12}$$

We proceed to the evaluation of the coefficients $a_0, a_1, a_2, \ldots$. As $d\xi/du$ may be regarded as a function of either ξ or u, let

$$d\xi/du = \psi(u) = \chi(\xi).$$

Then

$$a_\tau = \tfrac{1}{\tau!} \psi^{(\tau)}(0). \tag{13}$$

Since u and ξ vanish together, the derivatives of ψ at the point $u = 0$ can be calculated from the derivatives of χ at the point $\xi = 0$. The following method of procedure is perhaps the simplest. Let

$$v(\xi) = \frac{H - f(q)}{\xi^2} = \alpha + \beta\xi + \gamma\xi^2 + \ldots \tag{14}$$

Then $u = \xi \sqrt{v}$ and $\frac{du}{d\xi} = \frac{v + \frac{1}{2}\xi v'}{\sqrt{v}}$.

Let $$w = 2v(\xi) + \xi v'(\xi) = -\frac{f'(\xi + d)}{\xi}. \tag{14'}$$

Then $$\chi(\xi) = \frac{1}{\frac{du}{d\xi}} = \frac{2\sqrt{v(\xi)}}{w(\xi)}. \tag{15}$$

Differentiating,

$$\left.\begin{aligned} \psi'(u) &= \chi'(\xi)\,\chi(\xi) \\ \ldots\ldots &= \ldots\ldots\ldots \\ \psi^{(n)}(u) &= \chi(\xi)\,\frac{d}{d\xi}\left\{\psi^{(n-1)}(u)\right\} \\ \ldots\ldots &= \ldots\ldots\ldots \end{aligned}\right\} \tag{16}$$

In some cases these successive derivatives of ψ are simple and easily calculated functions of ξ. In others the successive derivatives rapidly become complicated. If $f(q)$ is given as a power series in ξ the process of differentiation can be performed conveniently as follows:

$$\left.\begin{aligned}
\psi'(u) &= \frac{2v'}{w^2} - \frac{4vw'}{w^3}, \\
\psi''(u) &= \frac{4\sqrt{v}}{w^3}\left\{ v'' - \frac{4v'w'}{w} - \frac{2vw''}{w} + \frac{6v(w')^2}{w^2} \right\} \\
\psi'''(u) &= \frac{4}{w^4}\left\{ v'v'' + 2vv''' - \frac{14vv''w'}{w} - \frac{4(v')^2w'}{w} - \frac{14vv'w''}{w} \right. \\
&\quad \left. + \frac{50vv'(w')^2}{w^2} + \frac{40v^2w'w''}{w^2} - \frac{4v^2w'''}{w^2} - \frac{60v^2(w')^3}{w^2} \right\}.
\end{aligned}\right\} \quad (17)$$

. . . . . .

At the point

$$\left.\begin{aligned}
&u = \xi = 0, \\
&w = 2v = 2\alpha, \qquad w' = 3v' = 3\beta \\
&w'' = 4v'' = 8\gamma \qquad \ldots\ldots\ldots \\
&w^{(n)} = (n+2)\,v^{(n)} = \ldots\ldots\ldots
\end{aligned}\right\} \quad (18)$$

Hence

$$\left.\begin{aligned}
\psi(0) &= \frac{1}{\sqrt{\alpha}} \\
\psi''(0) &= \frac{3}{\alpha^{3/2}}\left\{ \frac{5}{4}\frac{\beta^2}{\alpha} - \gamma \right\}, \\
\psi^{IV}(0) &= \frac{105}{\alpha^{7/2}}\left\{ \frac{33}{16}\frac{\beta^4}{\alpha^2} - \frac{9}{2}\frac{\beta^2\gamma}{\alpha^2} + \frac{2\beta\delta}{\alpha} + \frac{\gamma^2}{\alpha} - \frac{4}{7}\epsilon \right\}.
\end{aligned}\right\} \quad (19)$$

Combining the above equations with (13), (11), and (12), we obtain

$$\begin{aligned}
J &= \pi\psi(o)H + 2\pi\varphi(2)\frac{\psi''(0)}{2!}H^2 + 24\varphi(4)\frac{\psi^{IV}(0)}{4!}H^3 + \ldots \\
&= \frac{\pi H}{\sqrt{\alpha}}\left\{ 1 + \frac{3}{8\alpha^2}\left[\frac{5}{4}\frac{\beta^2}{\alpha} - \gamma\right]H + \frac{35}{64\alpha^3}\left[\frac{33\beta^4}{16\alpha^2} - \frac{9}{2}\frac{\beta^2\gamma}{\alpha^2}\right.\right. \\
&\qquad \left.\left. + \frac{2\beta\delta}{\alpha} + \frac{\gamma^2}{\alpha} - \frac{4}{7}\epsilon\right]H^2 + \ldots\ldots \right\}.
\end{aligned} \quad (20)$$

This result is similar in form to Tank's and the coefficients of the two lowest powers of H are the same as in his series, but the coefficient of H^3 is different.

In order that the result may be valid, it is necessary that the series (4) converge and represent $f(q)$ throughout the interval of integration, and that the derivatives of ψ shall all be finite throughout a circle whose radius is greater than $+H^{1/2}$ drawn about the origin on the complex u plane. From (17) it is evident that the derivatives of ψ will be finite up to the point where w vanishes. It follows from (14′) that $f'(q)/(q-d)$ must have no zeros on that part of the complex q plane which is mapped on the above-mentioned circle on the u plane. It is necessary, in particular, that $f'(q)$ shall vanish not more than once for real values of q between a and b.

It does not seem worth while to attempt an exact discussion of the boundaries of the region on the complex q plane from which the zeros of $f'(q)/(q-d)$ must be excluded, but we can say qualitatively that there is little chance that the series (7) will not converge throughout the interval of integration if the series

$$w = 2\alpha + 3\beta\xi + 4\gamma\xi^2 + 5\delta\xi^3 + \ldots \tag{21}$$

converges rapidly.

In the applications of this analysis an expression for H as a function of J will generally be desired. The power series may be reversed to advantage by the following scheme which resembles that of Lagrange. Let $H = F(J)$. Then by Taylor's theorem

$$H = JF'(0) + \frac{J^2}{2!}F''(0) + \frac{J^3}{3!}F'''(0) + \ldots. \tag{22}$$

Let b_n denote the coefficient of H^n in (20). $F'(0)$, $F''(0)$, etc., can be calculated in terms of the b's. Let us first compute the derivatives of F in terms of H. Let

$$p(H) = \sum_{n=1}^{\infty} nb_nH^{n-1} = \frac{dJ}{dH}.$$

Then

$$\frac{dH}{dJ} = F'(J) = \frac{1}{p(H)}. \tag{23}$$

Differentiating again,

$$\left.\begin{aligned} F''(J) &= \frac{d}{dH}\left(\frac{1}{p}\right)\cdot\frac{dH}{dJ} = -\frac{p'}{p^3} \\ F'''(J) &= \frac{d}{dH}\left(-\frac{p'}{p^3}\right)\frac{dH}{dJ} = -\frac{p''}{p^4} + \frac{3(p')^2}{p^5} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{24}$$

Since J vanishes when H does, the values of $F'(0)$, $F''(0)$, etc., are obtained by setting H equal to zero in the right-hand members of (23) and (24). Thus

$$\left.\begin{aligned} &F'(0) = 1/b_1; \qquad F''(0) = -2b_2/b_1^3; \\ &F'''(0) = -6b_3/b_1^4 + 12b_2^2/b_1^5; \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{25}$$

If these formulas for $F'(0)$, $F''(0)$, etc., are evaluated in terms of the expressions for the b's given by (20), equation (22) becomes

$$H = \frac{\sqrt{\alpha}}{\pi}J + \frac{3}{8\pi^2\alpha}\left[\gamma - \frac{5\beta^2}{4\alpha}\right]J^2 + \frac{1}{32\pi^3\alpha^{3/2}}\left[10\epsilon - \frac{35\beta\delta}{\alpha}\right.$$

$$-\frac{17}{2}\frac{\gamma^2}{\alpha}+\frac{225}{4}\frac{\beta^2\gamma}{\alpha^2}-\frac{705}{32}\frac{\beta^4}{\alpha^3}\Big]J^3+\ldots\ldots \qquad (26)$$

As a check on the series development here suggested, the writer has derived Sommerfeld's formula,

$$\oint\sqrt{A+\frac{2B}{q}+\frac{C}{q^2}}\,dq=2\pi\sqrt{-C}\left[\frac{B}{\sqrt{AC}}-1\right] \qquad (27)$$

by means of the series (12). The derivatives of ψ when evaluated by equations (16) are in this case particularly simple and as a result the series can be summed.

It is of interest to note that the method of development in series here suggested is applicable to a variety of problems. It may be used to evaluate indefinite as well as definite integrals.

One simple application is in the determination of the periods of oscillation of a vibrating system. Iu the case of a conditionally periodic system with orthogonal coördinates, the periods are given by expressions of the form

$$T=\oint\frac{dq}{\dot{q}}=2\int_a^b\frac{dq}{\sqrt{f(q)}}, \qquad (28)$$

where a and b are again roots of $f(q)$. Introducing the quantities H, ξ, u defined as in the preceding work, we obtain

$$T=2\int_{-\sqrt{H}}^{+\sqrt{H}}\frac{d\xi/du}{\sqrt{H-u^2}}\,du. \qquad (29)$$

If $d\xi/du$ is developed into a power series of the type (7), we obtain

$$T=\sum_{\tau=0}^{\infty}a_\tau\overline{K}_\tau$$

where

$$\overline{K}_\tau=2\int_{-\sqrt{H}}^{+\sqrt{H}}\frac{u^\tau\,du}{\sqrt{H-u^2}}=H^{\tau/2}\int_0^{2\pi}\sin^\tau\theta\,d\theta.$$

The coefficients $\overline{K}_\tau$ vanish when τ is odd, and the final expression for the period is

$$T=2\pi\left[a_0+2\sum_{\tau=1}^{\infty}a_{2\tau}(\tau+1)\,\varphi(2\tau)\,H^\tau\right]. \qquad (30)$$

[1] This is in effect the method used by Sommerfeld, *Atombau und Spektrallinien*, 2nd edition, Braunschweig, 1921, pp. 476–482.

[2] F. Tank, *Mitteilungen Physik. Ges. Zurich*, 1919, p. 87.

[3] Cf. G. Hettner, *Zs. Physik.*, 1, 4, 1920 (350).

THE FURTHER EXTENSION OF THE ULTRA-VIOLET SPECTRUM AND THE PROGRESSION WITH ATOMIC NUMBER OF THE SPECTRA OF LIGHT ELEMENTS

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The chief purpose of this investigation, outlined and begun in 1916[1] and briefly reported upon last year,[2] has been to explore in the extreme ultra-violet the radiations which can be emitted by the second ring or shell of electrons in the atoms of atomic number from 2 to 13 (helium to aluminium).[3]

The results obtained to date may be very briefly summarized as follows:[3]

1. The ultra-violet spectrum has been photographed and its wave-lengths determined down to $\lambda = 136.6$ Ångstroms in the case of aluminium and down to $\lambda = 149.5$ Ångstroms in the case of copper. There is thus a gap represented by a factor of but 10 between the shortest measured ultra-violet waves and the longest X-rays measured by the method of crystal-spectrometry which stops at 13.3 Ångstroms. *Fortunately, however, in the ultra-violet region, which has been already opened up and explored, the most interesting and the most important of the hitherto inaccessible radiations are found;*—thus

2. The L_α lines of Al, Mg, and Na have been photographed and located at 144.3 Å, 232.2 Å, and 372.2 Å, respectively. These wave-lengths are all fairly accurately on the Mosely line connecting L_α frequencies and atomic number (see fig. 1). *It has thus been definitely proved that the L series continues with its main characteristics unchanged throughout the whole range of atomic numbers from Ur (92) to Ne (10).* The linear progression thus revealed clear down to neon could be very roughly inferred from the beautiful measurements of Hjalmar,[4] which gave the K_α and K_β lines of the elements down to sodium. For the Kossel relation between the frequencies of the K and L series, namely, $K_\beta - K_\alpha = L_\alpha$, although known to be quite inexact, was presumably sufficiently near to the truth to enable *the order of magnitude* of the L_α wave-lengths to be predicted. Kossel has already made this use of Hjalmar's data.[5] His values of L_α computed for Al, Mg and Na in this way are actually about 20% too low.

3. It has been found that *the aluminium atom (atomic number 13) when excited by these condensed sparks in vacuo emits no radiations whatever of wave-length between 144.3 Å and about 1200 Å where its M spectrum,* that due to its three outer electrons, begins and extends with considerable complexity into the visible. *This shows that optical spectra are quite like*

X-ray spectra in that large gaps occur between the frequencies due to the electrons in successive rings or shells. If we could neglect the influence of the negative electrons upon one another we could compute the relative diameters of these shells, for they would be inversely proportional to the limiting frequencies, i.e., in this case in about the ratio 1 to 9; but such computations are of little value save as the roughest sort of indices. The chief lines below 2000 Å, due to the three outer electrons of the Al atom have the wave-lengths 1379.7 Å, 1384.5 Å, 1605.9 Å, 1612.0 Å, 1671.0 Å, 1854.7 Å and 1862.7 Å.

Magnesium shows a behavior quite like that of aluminium in that we find a complete blank between its L_α line at 232.2 Å and the lines due to its two outer electrons whose radiations begin on our plates on the short wave-length side at about 1700 Å. The chief radiations below 2000 Å, arising from these two outer electrons of the magnesium atom, have the wave-lengths 1735.2 Å, 1737.9 Å, 1751.0 Å, and 1753. 7 Å.

Also, *sodium*, quite consistently with the foregoing, is found to emit no radiations whatever between its L lines, the longest of which is at 376.5 Å and the lines due to its single M electron which have their convergence wave-length at 2412.63 Å, and reach their maximum intensity in the familiar sodium doublet at 5890 Å and 5896 Å.

4. Coming now to the group of atoms below neon, atomic number 10, the spectra due to the electrons in the incomplete second or L-ring in the case of these atoms are completely unpredictable from any theory that we now have but they have been experimentally obtained. For reasons which will appear they will not be considered in the order of atomic number.

a. The spectrum due to the six L-ring electrons of the oxygen atom (atomic number 8) begins upon our plates at about 230 Å and extends with much complexity and strength up to 834.0 Å where the strongest oxygen line, which will be arbitrarily called its L_α line, is found. Above 834.0 Å the oxygen lines are few in number and relatively faint.

Since the K_α line of O can be computed with much certainty to be at 23.68 Å, the ratio of the K frequency to the L frequency for the oxygen atom is about 35. It will be recalled that this ratio is only about 7 in the case of atoms of high atomic number and that it slowly increases with decreasing atomic number, reaching the values 17.2, 23.4 and 31.1 in the cases of Al, Mg, and Na, respectively (see above). The strongest of the oxygen lines have the following wave-lengths: 321.2, 374.3, 507.8, 525.7, 554.2, 599.5, 610.1, 616.7, 625.2, 629.6, 644.0, 703.1, 718.5 and 834.0.

The oxygen lines have been identified because they appear as impurities in all easily oxidizable metals, NrLi, Mg, Al, Zn, Fe, etc., The key to the oxygen spectrum was furnished by the discovery that chemically pure aluminium and magnesium showed the extraordinary property

of having identical spectra in all particulars between 1000 Å and 235 Å, while showing altogether distinct characteristic spectra between 1700 Å and 6000 Å, a behavior which could only mean that they themselves had no lines in the region, between 235 Å and 1000 Å, all of the observed lines being those of the common impurity oxygen—an interpretation supported by the fact that the well known oxygen lines of the visible region were also found in both spectra. It was with the aid of this discovery and the careful comparison of the spectra of otherwise pure metals, or of metals containing known impurities that the origins of practically all of the hundreds, even thousands, of new lines have been established with but very little uncertainty.

b. The spectrum due to the 4 *L*-ring electrons of the *carbon* atom begins upon our plates at 360.5 Å and extends with much complexity and strength up to 1335.0 Å where its strongest line, which, in harmony with the foregoing convention will be called its L_α[6] line is found. Above 1335.0 Å, the carbon lines are widely scattered and relatively weak. Since the *K* line of carbon is accurately computed to be at 44.4 Å, the ratio of the *K* to the *L* frequency in carbon is about 30.

It is interesting that practically the whole group of lines which Lyman obtained from his *condensed discharge in helium*[7] are due to carbon, oxygen, nitrogen and hydrogen. Thus those whose wave-lengths he gives as 599.0, 643.7, 702.9, 718.2, 796.8 and 834.1 are all due to oxygen; those whose wave-lengths he gives as 904.6, 977.2, 1010.6, 1037.0, 1175.9 and 1247.9 are due to carbon; those whose wave-lengths he gives as 916.7, 991.1 and 1085.5 are due to nitrogen; while those whose wave-lengths he gives as 972.7, 1026.0, 1216.0, are due to hydrogen.

c. The spectrum due to the 5 *L*-ring electrons of the nitrogen atom (atomic number 7) was obtained by using Al electrodes with ammonium nitrate in their cores and observing the new lines which were not due to aluminium, oxygen or hydrogen. Before this experiment was tried, it was predicted that the L_α line of nitrogen would have to lie between the L_α line of oxygen and that of carbon. It was found that the spectrum due to the nitrogen atom was very simple, beginning on our plates on the short wave-length side at 685.6 Å and *reaching a maximum in the line of wave-length 1085.3 Å, precisely as predicted.* This 1085.3 line is taken, in accordance with the foregoing convention, as the L_α line of the nitrogen atom. The only other strong lines which we have obtained which are due to this atom have the following wave-lengths: 685.6, 916.2, 991.1. The pair of nitrogen lines found by Lyman at 1492.8 and 1494.8 appear upon our plates, but they are very faint in comparison with the foregoing lines, the nitrogen spectrum thus showing a behavior quite like that due to the atoms of oxygen and carbon. Since the K_α line of nitrogen is at 31.2 Å, the ration of the K_α to the L_α frequency is 34.8.

d. In the case of *fluorine* a strong line has been found through the use of sodium fluoride at 657.2 Å. This is in about the position predicted by the foregoing mode of approach for its L_α line. Another strong fluorine line appears at 607.2 Å. These are the only lines thus far definitely identifiable as coming from the 7 L- ing electrons of the fluorine atom. The longer of these wave-lengths is taken as the L_α line of fluorine. Fluorine probably has, however, other lines of shorter wave-lengths but of such intensity that we have not yet been able to obtain them.

e. The spectrum, due to the 3 L-ring electrons of boron (atomic number 5) is especially interesting because of its simplicity. It contains less than 10 strong lines all told. It begins on the short wave-length side at 676.8 Å and has only the following strong lines: 676.8, 760.0, 1624.4, the two doublets at 2164.2, 2166.2 and 2496.9, 2497.8 and the single spark-line near the visible at 3451.5. According to the foregoing convention, its L_α line should be the doublet at 2497 Å. Since the K_α line of boron is at 67.2 Å, the ratio of the K_α to the L_α frequency in boron is about 37.

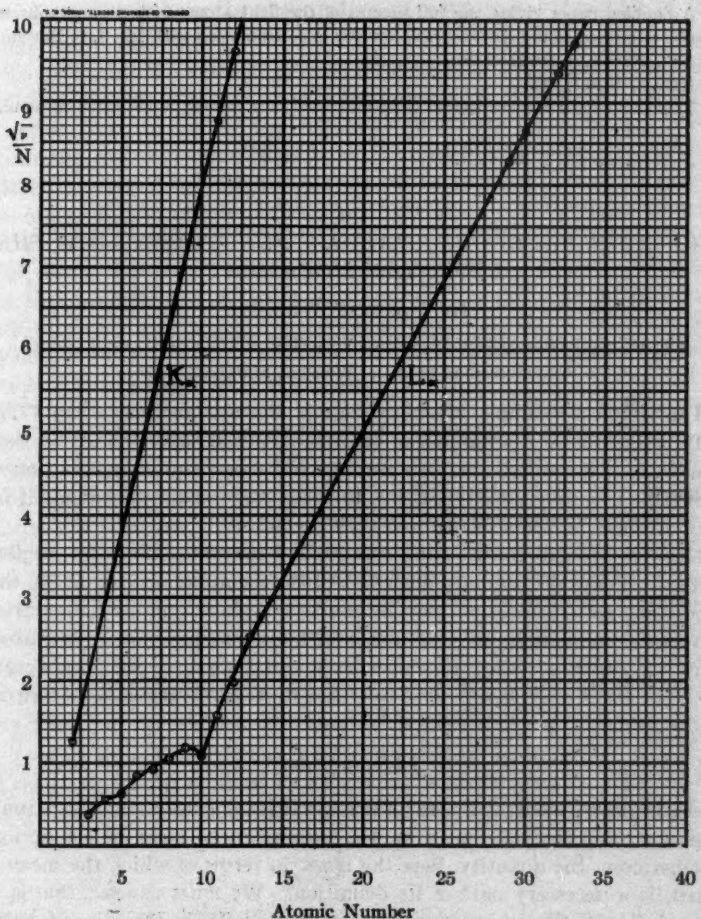
f. The spectrum due to the 2 L-ring electrons of *beryllium* begins on the short wave-length side, according to all the data available up to date, at 2175 Å and reaches its maximum, its L_α doublet, at 3130.6 and 3131.2. The entire absence upon our plates of any lines whatever due to beryllium between 230 Å and 2100 Å is a conspicuous illustration of the wide gaps in spectra obtained with ordinary gratings in or near the visible region.

g. Similiarly the present experiments with *lithium* (atomic number 3) reveal no lines whatever between the shortest wave-lengths measurable on the plates used in the case of lithium and the familiar series due to its single L-ring electron whose L_α line is at 6708 Å and whose convergence wave-length is at 2299 Å.

5. The graph of the L_α lines of the elements from lithium up is shown in the accompanying figure, along with the corresponding graph for the K_α lines from helium up taken from the work of others. *The only point on the L_α graph which has not been directly observed is that corresponding to neon which has been inserted from the resonance potential of that gas.*

It is to be observed that the method here employed gives, as I think, the characteristic spectra of the *atoms* of each element, not of the molecules. If it were possible to work with the *atomic gas* of each element, the L_α lines given herewith would be the *resonance* potentials of these atoms.

The progression thus revealed in these optical spectra is exceedingly interesting and simple, and very like that exhibited by X-ray spectra. The reason it has not been observed before is clearly because hitherto only the upper ends of these optical spectra have been observable, so that the unfolding of simple relationships between spectra and atomic number had to await the development of an ultra-violet technique.



L_α lines of Li, Be, B, C, N, O, F, Ne, Na, Mg and Al

A glance at the figure shows that we now have the complete outline of all of the types of radiations which are emitted by atoms of small atomic number. It remains only to fill in the details of their K_α and L_α spectra. One interesting fact which appears from a glance at the whole series of spectra of elements from hydrogen to neon is that atoms of odd ordinal all appear to have simple, few-lined spectra, while those of even ordinal number have much more complex spectra.

[1] *Physic. Rev.*, **10**, 1917 (205).
[2] *Astroph. J.* **52**, 1920 (47).

[3] A detailed paper giving the full spectra of the light atoms with photographs will soon be published in the *Astrophysical Journal* in collaboration with Mr. I. S. Bowen.

[4] *Zeit. Physik*, **1**, 1920 (439).

[5] *Ibid.*, **2**, 1920 (470).

[6] This convention is more logical than that used in a former paper (cit.[2]) in naming the L_α line.

[7] *Astroph. J.*, **43**, 1916 (102).

THE AVERAGE OF AN ANALYTIC FUNCTIONAL AND THE BROWNIAN MOVEMENT

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The simplest example of an average is the arithmetical mean. The arithmetical mean of a number of quantities is their sum, divided by their number. If a result is due to a number of causes whose contribution to the result is simply additive, then the result will remain unchanged if for each of these causes is substituted their mean.

Now, the causes contributing to an effect may be infinite in number and in this case the ordinary definition of the mean breaks down. In this case some sort of measure may be used to replace number, integration to replace summation, and the notion of mean reappears in a generalized form. For instance, the distance form one end of a rod to its center of gravity is the mean of its length with reference to its mass, and may be written in the form

$$\int l dm \div \int dm$$

where l stands for length and m for mass. It is to be noted that l is a function of m, and that the mean we are defining is the mean of a function. Furthermore, the quantity, here the mass, in terms of which the mean is taken, is a necessary part of its definition. We must assume, that is, a normal distribution of some quantity to begin with, in this case of mass.

The mean just discussed is not confined to functions of one variable; it admits of an obvious generalization to functions of several variables. Now, there is a very important generalization of the notion of a function of several variables: the function of a line. For example, the attraction of a charged wire on a unit charge in a given position depends on its shape. The length and area of a curve between two given ordinates depend on its shape. As a curve is essentially a function, these functions of lines may be regarded as functions of functions, and as such are known as functionals. Since a function is determined when its value is known for all arguments, a functional depends on an infinity of numerical determi-

nations, and may hence be regarded as in some wise a function of infinitely many variables.

To determine the average value of a functional, then seems a reasonable problem, provided that we have some convention as to what constitutes a normal distribution of the functions that form its arguments. Two essentially different discussions have been given of this matter: one, by Gâteaux, being a direct generalization of the ordinary mean in n-space;[1] the other, by the author of this paper,[2] involving considerations from the theory of probabilities. The author assumes that the functions $f(t)$ that form the arguments of his functionals have as their arguments the time, and that in any interval of the small length ϵ as many receive increments of value as decrements of equal size. He also assumes that the likelihood that a particle receive a given increment or decrement is independent of its entire antecedent history.

When a particle is acted on by the Brownian movement, it is in a motion due to the impacts of the molecules of the fluid in which it is suspended. While the retardation a particle receives when moving in a fluid is of course due to the action of the individual particles of the fluid, it seems natural to treat the Brownian movement, in a first approximation, as an effect due to two distinguishable causes: (1) a series of impacts received by a particle, dependent only on the time during which the particle is exposed to collisions; (2) a damping effect, dependent on the velocity of the particle. If we consider one component of the total impulse received by a particle under heading (1), we see that it may be considered as a function of the time, and that it will have the sort of distribution which will make our theory of the average of a functional applicable.

It will result directly from the previous paper of the author that if $f(t)$ is the total impulse received by a particle in a given direction when the unit of time is so chosen that the probability that $f(t)$ lie between a and b is

$$\frac{1}{\sqrt{\pi i}}\int_a^b e^{-\frac{x^2}{i}}\,dx$$

then the average value of

$$A + \int_a^b f(t)\,G(t)dt + \int_a^b\int_a^b f(s)\,f(t)\,H(s,t)\,ds\,dt \quad [H(s,t) = H(t,s)]$$

will be

$$A + \int_a^b\int_a^{\cdot} t\,H(s,t)\,ds\,dt. \qquad (1)$$

We now proceed to a more precise and detailed treatment of the question.

2. Einstein[3] has given as the formula for the mean square displacement in a given direction of a spherical particle of radius r in a medium of viscosity η over a time t, under the action of the Brownian movement, the formula

$$\overline{d_t^2} = RTt \div 3\pi r\eta N$$

where R is the gas-constant, T the absolute temperature of the medium, and N the number of molecules per gram-molecule. In the deduction of this formula, Einstein makes two important assumptions. The first is that Stokes' law holds concerning forces of diffusion. Stokes' law states that a force F will carry particle of radius r through a fluid of viscosity η with velocity $F \div 6\pi r\eta$. Einstein's second assumption is that the displacement of a particle in some interval of time small in comparison with those which we can observe is independent, to all intents and purposes, of its entire antecedent history. It is the purpose of this paper to show that even without this assumption, under some very natural further hypotheses, the departure of $\overline{d_t^2}/t$ from constancy will be far too small to observe.

In this connection, it is well to take note of just what the Brownian movement is, and of the precise sense in which Stokes' law holds of particles undergoing a Brownian movement. In the study of the Brownian movement, our attention is first attracted by the enormous discrepancy between the apparent velocity of the particles and that which must animate them if, as seems probable, the mean kinetic energy of each particle is the same as that of a molecule of the gas. This discrepancy is of course due to the fact that the actual path of each particle is of the most extreme sinuosity, so that the observed velocity is almost in no relation to the true velocity. Now, Stokes' law is always applied with reference to movements at least as slow as the microscopically observable motions of a particle. It hence turns out that Stokes' law must be treated as a sort of average effect, or in the words of Perrin,[4] "When a force, constant in magnitude and direction, acts in a fluid on a granule agitated by the Brownian movement, the displacement of the granule, which is perfectly irregular at right angles to the force, takes in the direction of the force a component progressively increasing with the time and in the mean equal to $Ft \div 6\pi\zeta a$, F indicating the force, t the time, ζ the viscosity of the fluid, and a the radius of the granule."

It is a not unnatural interpretation of this statement to suppose that we may assume the validity of Stokes' law for the slower motions which are all that we see directly of the Brownian motion, so that we may regard the Brownian movement as made up (1) of a large number of very brief, independent impulses acting on each particle and (2) of a continual damping action on the resulting velocity in accordance with Stokes' law. It is to be noted that the processes which we treat as impulsive forces need not be the simple results of the collision of individual molecules with the particle, but may be highly complicated processes, involving intricate interactions between the particle and the surrounding molecules. It may readily be shown by a numerical computation that this is the case.

It follows from considerations discussed at the beginning of my paper on *The Average of an Analytic Functional* that after a time the probabil-

ity that the total momentum acquired by a particle from the impacts of molecules will lie between x_0 and x_1 is of the form

$$\frac{1}{\sqrt{\pi ct}}\int_a^b e^{-\frac{x^2}{ct}}dx.$$

Superimposed on this momentum is that due to the viscosity acting in accordance with Stokes' law, namely $6\pi r\eta V$, where V is the velocity of the particle. We shall write Q for $6\pi r\eta \div M$, where M is the mass of a particle.

Let us write τ for ct. Let the total impulse received by a given particle in time t, neglecting the action of viscosity, be $f(\tau)$. Consider $m(t)$, the actual momentum of the particle, as a function of t. Then

$$m(t+dt) = m(t) + f(ct+cdt) - f(ct) - Qm(ct+c\vartheta dt)dt \quad (o \leq \vartheta \leq 1).$$

We cannot treat this as a differential equation, as we have no reason to suppose that f has a derivative. We can make it into an integral equation, however, which will read

$$m(t) - m(o) = f(ct) - Q\int_o^t m(t)dt.$$

Clearly one solution of this integral equation is

$$m(t) = m(o)e^{-Qt} + f(ct) - Qe^{-Qt}\int_o^t f(ct)e^{Qt}dt,$$

and there is no difficulty in showing that an integral equation of this sort can have only one continuous solution.

Another integration gives for the distance traversed by the particle in time t

$$\begin{aligned} d_t &= \frac{1}{M}\left\{\left[-\frac{m(o)e^{-Qt}}{Q}\right]_o^t + \int_o^t f(ct)dt - \int_o^t Qe^{-Qt}\left[\int_o^t e^{Qt}f(ct)dt\right]dt\right\} \\ &= \frac{1}{M}\left\{\frac{m(o)}{Q}(1-e^{-Qt}) + e^{-Qt}\int_o^t e^{Qt}f(ct)dt\right\} \qquad (2) \\ &= \frac{1}{M}\left\{\frac{m(o)}{Q}(1-e^{-Qt}) + \frac{e^{-Qt}}{c}\int_o^\tau e^{\frac{Q\tau}{c}}f(\tau)d\tau\right\} \end{aligned}$$

Applying the methods of my previous paper, we get for the mean value of d_t^2, in accordance with (1);

$$\begin{aligned} \overline{d_t^2} &= \frac{1}{M^2}\left[\frac{m(o)}{Q}(1-e^{-Qt})\right]^2 + \frac{1}{M^2c^2}e^{-2Qt}\int_o^{ct}\int_o^{ct} A\{f(x)f(y)\}e^{\frac{Q(x+y)}{c}}dydx \\ &= \frac{1}{M^2}\left[\frac{m(o)}{Q}(1-e^{-Qt})\right]^2 + \frac{1}{M^2c^2}e^{-2Qt}\int_o^{ct}\left[\int_o^x ye^{\frac{Q(x+y)}{c}}dy\right]dx \\ &= \left[\frac{m(o)}{MQ}(1-e^{-Qt})\right]^2 + \frac{e^{-2Qt}}{M^2c^2}\int_o^{ct}\left[\frac{cx}{Q}e^{\frac{2Qx}{c}} - \frac{c^2}{Q^2}e^{\frac{2Qx}{c}} + \frac{c^2}{Q^2}e^{\frac{Qx}{c}}\right]dx \qquad (3) \\ &= \left[\frac{m(o)}{MQ}(1-e^{-Qt})\right]^2 + \frac{c}{4M^2Q^3}\left[2Qt - 3 + 4e^{-Qt} - e^{-2Qt}\right] \end{aligned}$$

Therefore

$$\left|\overline{d_i^2}/t - c/(2M^2Q^2)\right| \leq 1/t\left[m(o)(1-e^{-Qt})/(MQ)\right]^2 + c(3-e^{-Qt})(1-e^{-Qt})/(4M^2Q^3t)$$

$$\leq \left[m(o)/(MQ)\right]^2 + 3c/(4M^2Q^3)$$

This represents the absolute departure of $\overline{d_i^2}/t$ from constancy. Writing v for $m(o)/M$, the initial velocity of the particle, we get

$$\frac{\left|\overline{d_i^2}/t - c/(2M^2Q^2)\right|}{c/(2M^2Q^2)} \leq \frac{v^2/Q^2}{c/2M^2Q^2} + 3/2Q$$

This is a measure of the relative departure of $\overline{d_i^2}/t$ from constancy. v cannot exceed, on the average, the velocity given on the average to the particle on the basis of the equipartition of energy; actually it is much smaller. $c/(2M^2Q^2)$ can be found directly, as it is nearly the observed value of $\overline{d_i^2}/t$. Q can be readily computed from the constants of the particles. Taking as a typical case one of Perrin's experiments on gamboge, Q turns out to be of the order of magnitude of 10^8, $c/2M^2Q^2$ of the order of magnitude of 10^{-8}, and the kinetic energy velocity of the order of magnitude of 10^{-1}. Hence the proportionate error is of the order of magnitude of 10^{-8}.

A proportionate error thus small is quite beyond the reach of our methods of measurement, so that we are compelled to conclude that d_i^2/t, under the hypotheses we have here formulated, is sensibly constant. There are cases, however, which seem to give a slightly different value of $\overline{d_i^2}/t$ for small values of the time than for larger values. The explanation has been suggested[5] that over small periods the Einstein independence of an interval on previous intervals does not hold. The result of the present paper would be to suggest strongly, if not to demonstrate, that the source of the discrepancy, if, as appears, it is genuine, and not due to experimental error, is in the fact that Stokes' law itself is only a rough approximation, and that the resistance does not vary strictly as the velocity.

[1] *Paris, Bull. Soc. Math. France*, 1919, pp. 47–70.

[2] "The Average of an Analytic Functional," in the last number of these PROCEEDINGS.

[3] *Leipzig, Ann. Physik*, 17, 1905 (549).

[4] *Ann. Chim. Phys.*, Sept., 1909; tr. by F. Soddy.

[5] Cf. Kleeman, *A Kinetic Theory of Gases and Liquids*, §§ 56, 60.

MEASUREMENTS OF THE DEVIATION FROM OHM'S LAW IN METALS AT HIGH CURRENT DENSITIES

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Any picture of the mechanism of current conduction in metals which takes account of the part played by the electrons would lead to the expectation of departures from Ohm's law at high current densities. On the classical free electron basis J. J. Thomson has shown that at currents of the order of 10^9 amp./cm^2. the current would be expected to increase as the square root of the applied E. M. F., and hence that the resistance will increase indefinitely. Many attempts have been made to detect the existence of this effect experimentally, but without success. The chief source of difficulty has been the necessity for separating the change of resistance due to the great temperature rise under the heavy current from the change due to a departure from Ohm's law. The best known attempt in this direction is perhaps that of Maxwell.[1] Assuming that the departure from Ohm's law must be proportional to the square of the current, which is plausible on grounds of symmetry, he showed that at a density of 1 amp./cm^2. the resistance of platinum, iron, and German silver does not differ by more than 1 part in 10^{12} from the resistance at infinitely small currents. His maximum density was about 5×10^4 amp./cm^2.

By the application of a new method I have been able to eliminate the source of error due to temperature rise, to detect the existence of the effect, and to measure it with a fair degree of accuracy. The specimen is made one of the arms of a bridge, and is traversed simultaneously by a heavy direct current and a small superposed alternating current of acoustical frequency. The resistance of the specimen to the direct current is measured with an ordinary galvanometer, and the resistance to the alternating current is measured at the same time with a telephone. If there is a departure from Ohm's law under the heavy current, that is if the relation between current and E. M. F. is not linear, the two resistances will not be equal, and from their difference the departure from Ohm's law may be calculated. The reason for

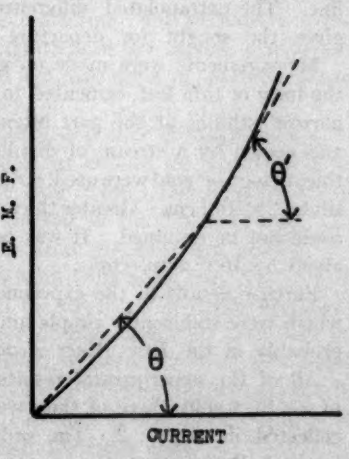

FIG. 1

this will be evident from an inspection of figure 1. It is to be remembered that a bridge is an instrument for balancing potentials in different parts of a net-work. At D. C. balance the potential drop corresponding to $\tan\theta$ in figure 1 is measured and at A. C. balance the potential drop corresponding to $\tan\theta'$.

It is evident that the direct effect of the unknown rise of temperature is eliminated by this method, because the temperature of the conductor is the same to both the direct and alternating current. There is, however, an indirect effect which is important, and which must be eliminated. Under the joint action of the direct and alternating current the wire receives a comparatively large supply of heat steadily, with a small heating and cooling effect superposed. Under this superposed heating and cooling the wire experiences alternations of temperature which give rise to fluctuations of resistance. The large direct current flowing through the fluctuating resistance gives rise to an alternating difference of potential in one of the bridge arms, which affects the A. C. balance. This action is like that of a microphone. It becomes vanishingly small at high frequencies of the alternating current, because the fluctuations of temperature become vanishingly small under these conditions.

The "microphone" effect was eliminated by making readings at a number of frequencies, plotting against the reciprocal of frequency and extrapolating to zero (that is, infinite frequency). The range of frequencies employed was from 320 to 3750 cycles per second. The extrapolation is therefore over a range only one-tenth of the observed readings. As further adding to the certainty of the extrapolation, it may be shown by a dimensional argument that the curve extrapolates to zero as a straight line. The extrapolated difference between D. C. and A. C. resistance gives the sought for departure from Ohm's law.

Measurements were made on gold and silver. These metals were in the form of thin leaf, cemented to a glass backing, cut into the shape of a narrow isthmus at the part intended to carry the high current density, and cooled by a stream of distilled water flowing over the glass. Two thicknesses of gold were used, 8×10^{-6} and 1.67×10^{-5}, and one thickness of silver, 2×10^{-5} cm. Greater thicknesses of gold were tried, but good results could not be obtained. It was possible to reach current densities up to about 5×10^{-6} amp./cm^2.

Further details of the experiment and of the electrical arrangements, which were sufficiently simple and obvious, will be described elsewhere, probably in the *Proc. Amer. Acad. Arts. and Sci.*

All of the experimental results obtained on eleven different samples of 8×10^{-6} gold (three of these were films formed by cathode deposit) are collected in figure 2. The ordinates are the extrapolated difference between D. C. and A C. resistance in terms of the initial resistance, and

the abscissae are current densities in 10^6 amp./cm^2. The specimens varied in breadth from 0.06 to 0.22 mm. There was no correlation between the results and the breadth, although the microphone effect has a very

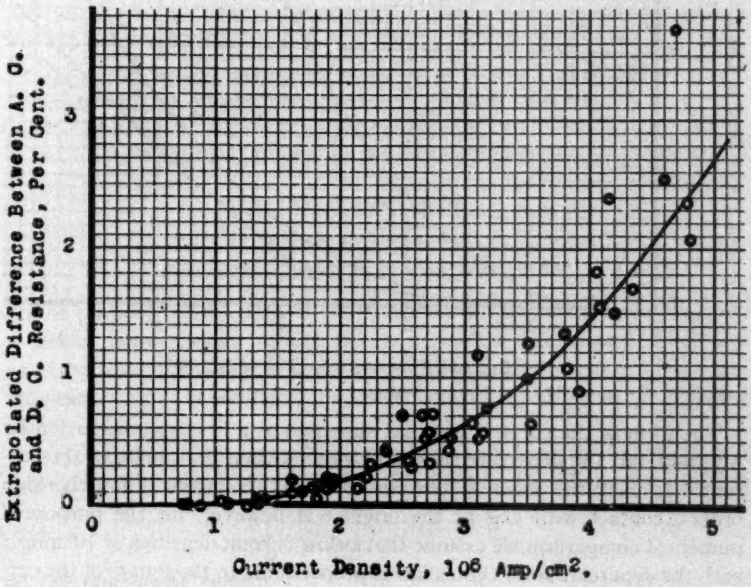

FIG. 2

strong connection. The results are scattering, but perhaps not more than would be expected when the magnitude of the current densities and the fact that different samples of gold leaf may differ in specific resistance by a factor of 2 or more is considered.

The curve drawn through the observed points is taken as the best mean of the experimental results. Let us denote the equation of this curve by $\varphi(x)$. Then the departure from Ohm's law is given by the expression $\int \varphi(x)dx/x$. This will be proved in the detailed paper. This integral may be calculated graphically from the observed points.

In figure 3 is given the departure from Ohm's law, calculated in this way, for gold of two thicknesses and for silver. The departure is positive, that is, the resistance is greater at high density. For thin gold and silver the departure rises to something of the order of 1% at a density of 5×10^{-6} amp./cm^2; for the thicker gold it is greater. It was not possible to reach as high current densities in the thick as the thinner gold. The accuracy is greatest for thin and least for thick gold.

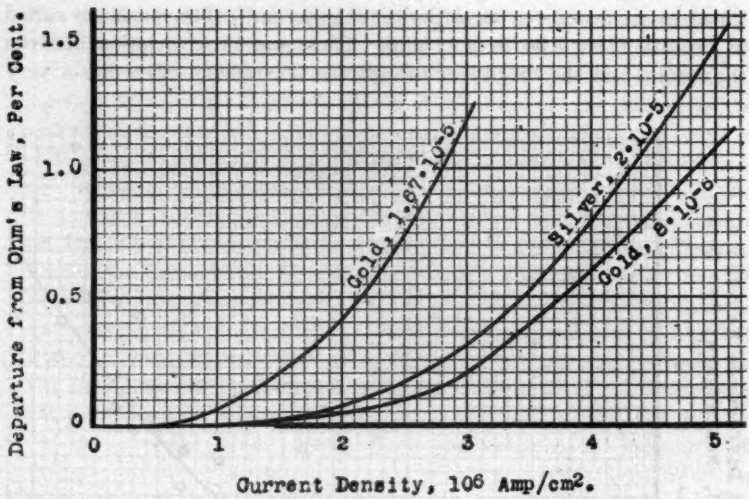

FIG. 3

The form of the curve is not that supposed as a first approximation by Maxwell, but the departure rises more rapidly than the square of the current. In fact the form of the curves seems to suggest infinitely high order of contact with axis at the origin. If however, for the purpose of numerical comparison we assume that below current densities of 10^6 amp./cm². the departure from Ohm's law is proportional to the square of the current within the limits of error, than the curve for thin gold shows that for it the resistance at 1 amp./cm². cannot differ by more than 1 part in 10^{16} from the resistance at infinitely small density (against 1 part in 10^{12} of Maxwell).

Theoretically the existence of this effect at currents of the order of 10^6 amp./cm². means that, granted a free path mechanism of conduction at all, the free path must be much longer than supposed on the classical basis, and accordingly the number of free electrons much less. Now on other grounds I have been led to the belief that conduction is by means of a free path mechanism,[2] of a different sort than that supposed in the classical theory, and that the number of electrons is much less and their paths longer than according to that theory. These new results are now in accord with this point of view. I have not been able as yet to carry through a more exact discussion on the free path basis and to obtain the numerical value for the length of the free path which these results would involve.

The difference in results for gold of different thickness is also what would be expected. The results for thick gold are not nearly as accurate as for the thinner, but there can seem no question but that the departure is

really greater for thick than for thin gold. If the free path is of the order of 10^{-5} cm., that is, comparable with the thickness of the leaf, then the average path in the thicker metal will be longer than in the thinner, and a greater departure would be expected in the thicker metal, as found.

[I am much indebted to my assistant, Mr. J. C. Slater, for his skill in making the readings.]

[1] Maxwell, C., Everett, J. D. and Schuster, A., *B. A. Rep.* 1876 (36–63).
[2] Bridgman, P. W., *Physic. Rev., Ithaca*, (2), **17**, 1921 (161–194).

AN INTEGRAL EQUALITY AND ITS APPLICATIONS

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The purpose of this note[1] is to deduce an integral equality adjoined to a linear homogeneous differential equation of the second order and to show some applications of such equalities to the question of the distribution of zeros of solutions of such differential equations in the complex plane.

Let $G(z)$ and $K(z)$ be two single-valued and analytic functions of z throughout the region under consideration and take the differential equation

$$\frac{d}{dz}\left[K(z)\frac{dw}{dz}\right] + G(z)w = 0, \tag{1}$$

or the equivalent system

$$\left\{\begin{aligned} &\frac{dw_1}{dz} = \frac{w_2}{K(z)}, \\ &\frac{dw_2}{dz} = -G(z)w_1, \end{aligned}\right. \quad \text{with} \quad \left\{\begin{aligned} &w_1 = w, \\ &w_2 = K(z)\frac{dw}{dz}. \end{aligned}\right. \tag{2}$$

From this system we easily deduce the relation

$$[\bar{w}_1 w_2]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2 \overline{\left[\frac{dz}{K(z)}\right]} + \int_{z_1}^{z_2} |w_1|^2 G(z)dz = 0, \tag{3}$$

where $\bar{u}$ denotes the conjugate of u. If we put

$$dK = \frac{dz}{K(z)}, \quad d\Gamma = G(z)dz, \tag{4}$$

relation (3) becomes

$$[\bar{w}_1 w_2]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2 \, \overline{dK} + \int_{z_1}^{z_2} |w_1|^2 d\Gamma = 0, \tag{5}$$

or, split up into real and imaginary parts,

$$\mathfrak{Re}\,[\bar{w}_1 w_2]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2\, d\mathrm{K}_1 + \int_{z_1}^{z_2} |w_1|^2\, d\Gamma_1 = 0, \tag{6}$$

$$\mathfrak{Im}\,[\bar{w}_1 w_2]_{z_1}^{z_2} + \int_{z_1}^{z_2} |w_2|^2\, d\mathrm{K}_2 + \int_{z_1}^{z_2} |w_1|^2\, d\Gamma_2 = 0, \tag{7}$$

where

$$d\mathrm{K} = d\mathrm{K}_1 + i d\mathrm{K}_2;\ d\Gamma = d\Gamma_1 + i d\,\Gamma_2. \tag{8}$$

We call relation (5) *Green's transform* of the differential equation (1). This relation can be used in many ways for obtaining information concerning the distribution of the zeros of a function $w(z)$, satisfying (1).

Formula (5) enables us to assign regions, below called *zero-free domains* where there can be no zeros of $w(z)$ or $\frac{dw}{dz}$. Some of these ways are indicated below.

The four differential equations

$$d\mathrm{K}_1 = 0,\ d\mathrm{K}_2 = 0;\ d\Gamma_1 = 0,\ d\Gamma_2 = 0, \tag{9}$$

define four families of curves; the K_1-family and the K_2-family forming the K-net and the Γ_1 and Γ_2-families forming the Γ-net. The two families belonging to the same base net are orthogonal trajectories of each other.

Take a solution $w_a(z)$ of (1) such that

$$W_a(z) = K(z)\, w_a(z) \frac{dw_a}{dz}$$

vanishes at a regular point a in the complex plane. Construct the Riemann surface on which $w_a(z)$ is single-valued and mark the two base nets on the surface. Further, draw all curves on the surface, starting from $z = a$ which do not pass through any of the singular points of the differential equation and which are composed of arcs of curves of the two base nets such that along the whole path one and the same of the following four inequalities is fulfilled, namely

$$(10)\quad 1^\circ \begin{cases} d\mathrm{K}_1 \geq 0, \\ d\Gamma_1 \leq 0; \end{cases} \quad 2^\circ \begin{cases} d\mathrm{K}_1 \leq 0, \\ d\Gamma_1 \geq 0; \end{cases} \quad 3^\circ \begin{cases} d\mathrm{K}_2 \geq 0, \\ d\Gamma_2 \geq 0; \end{cases} \quad 4^\circ \begin{cases} d\mathrm{K}_2 \leq 0, \\ d\Gamma_2 \leq 0. \end{cases}$$

We agree to smooth the corners of the path-curves, when necessary, in order to preserve the continuity of the tangent along the curve. Such curves we call *standard paths* and designate the four different kinds (distinguished by their characteristic inequalities) respectively

$$S\mathrm{K}_1^+,\ S\mathrm{K}_1^-,\ S\mathrm{K}_2^+ \text{ and } S\mathrm{K}_2^-. \tag{11}$$

The points on the surface which belong to at least one standard path, emanating from $z = a$, together form the *standard domain* $D(a)$ of a. A

point on the boundary of $D(a)$ is counted part of the standard domain provided it is not a singular point of the differential equation and is different from a itself. In view of (6) and (7) we obtain the result:

There is no zero of $W_a(z)$ in the standard domain of a.

Here is another application of the Green's transform. Suppose $G(z)$ and $K(z)$ to be analytic and, furthermore, real on an interval (a,b) of the real axis. Then there are solutions of (1) which are real on the same interval. Draw all standard paths of the third and the fourth kind, SK_a^+ and SK_a^-, which emanate from the points on (a,b). The points on these paths, not including possible singular points and the points on (a,b), form a zero-free region for all solutions real on (a,b) in virtue of formula (7). For this statement we have only used the fact that for real solutions $\mathfrak{Im}[\overline{w_1(z)}\ w_2(z)] = 0$ on (a,b). The generalizations are obvious.

The Standard Domain Is Covariant under Conformal Transformation.— By this we understand that a change of independent variable from z to Z by putting $Z = F(z)$ which preserves the form of the differential system (2), carries the standard domain of a point z_0 in the z-plane over into the standard domain of the corresponding point Z_0 in the Z-plane and these two domains are the conformal maps of each other by the transformation $Z = F(z)$ and its inverse, provided $F(z)$ is regular in $D(z_0)$ and $F'(z) \neq 0$ there. This is obvious because Green's transform is an invariant under such a transformation and the base nets in the two planes correspond to each other by the conformal transformation.

The standard domain is not covariant under a simultaneous change of dependent and independent variable. By a fortunate choice of such variables it is often possible to determine zero-free domains of such an extent that there is comparatively little freedom left for placing possible zeros; thus one is able to obtain a fairly good qualitative description of the arrangement of the zeros in the plane. The transformation

$$(12)\quad \begin{cases} dZ = \sqrt{\dfrac{G(z)}{K(z)}}\, dz, \\ W = \sqrt{S(Z)}\, w, \\ S(Z) = \sqrt{G(z)K(z)}, \end{cases}$$

often yields good service for investigation of the distribution of zeros of solutions in the neighborhood of an irregular singular point of the differential equation.

[1] This note is an abstract of a paper offered to *Trans. Amer. Math. Soc.* for publication.